An Exposition of Spectral Graph Theory

Mathematics 634: Harmonic Analysis

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Abstract

Recent developments have led to an interest to characterize signals on connected graphs to better understand their harmonic and geometric properties. This paper explores spectral graph theory and some of its applications. We show how the 1-D Graph Laplacian, Δ_D, can be related to the Fourier Transform by its eigenfunctions which motivates a natural way of analyzing the spectrum of a graph. We begin by stating the Laplace-Beltrami Operator, Δ, in a general case on a Riemann manifold. The setting is then discretized to allow for formulation of the graph setting and the corresponding Graph Laplacian, Δ_G. Further, an exposition on the Graph Fourier Transform is presented. Operations on the Graph Fourier Transform, including convolution, modulation, and translation are presented. Special attention is given to the equivalences and differences of the classical translation operation and the translation operation on graphs. Finally, an alternate formulation of the Graph Laplacian on meshes, Δ_M, is presented, with application to multi-sensor signal analysis.

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1. Background and Motivation

Classical Fourier analysis has found many important results and applications in signal analysis. Traditionally, Fourier analysis is defined on continuous, $\mathbb{R}$, or discrete, $\mathbb{Z}$, domains. Recent developments have led to an interest in analyzing signals which originate from sources that can be modeled as connected graphs. In this setting the domain also contains geometric information that may be useful in analysis. In this paper we analyze methods of Fourier analysis on graphs and results in this setting motivated by classical results.
We begin with the classical Fourier Transform,

Definition 1.1. For a function $f(t), t \in \mathbb{R}$, we let the Fourier Transform, $\widehat{f}(\gamma)$, be defined as,

$$\widehat{f}(\gamma) = \int_{\mathbb{R}} f(t) e^{-2\pi i \gamma t} \mathrm{d}t = \langle f, e^{-2\pi i \gamma t} \rangle. \tag{1}$$

Further, in the discrete setting,

Definition 1.2. For a function $f[n], n \in \mathbb{Z}$, we let the Discrete Fourier Transform (DFT), $\widehat{f}[n]$, be defined as,

$$\widehat{f}[\gamma] = \sum_{n=1}^{N} f[n] e^{-2\pi i \gamma n/N} = \langle f, e^{-2\pi i \gamma t} \rangle \tag{2}$$

where we let $[\cdot]$ denote a function with discrete indices.

We now note the following relationship between the DFT and spectral analysis via the Laplacian, the DFT basis functions form a set of eigenvectors of the 1-D discrete Laplace Operator, Δ_D.
For some $x \in \mathbb{R}^N$ of finite length we say,

$$\Delta_D x = \frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i) \tag{3}$$

where the indices are incremented and decremented modulo N, the number of elements. This can be written in matrix form such that

$$\Delta_D x = -\boldsymbol{L}_D x \tag{4}$$

where

$$\boldsymbol{L}_D = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \dots & \dots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \dots & \dots & 0 & -\frac{1}{2} & 1 \end{bmatrix}. \tag{5}$$

So,

Lemma 1.1. *The basis vectors of the unitary DFT are orthonormal eigenvectors of any circulant matrix. Moreover, the eigenvalues of a circulant matrix are given by the DFT of its first column.*

Proof. Let $W_N = e^{-2\pi i/N}$ and $\mathbf{F} = \left\{ \frac{1}{\sqrt{N}} W_N^{kn}, 0 \le k, n \le N-1 \right\}$ be the $N \times N$ unitary DFT matrix. Let $\mathbf{H}$ be an $N \times N$ circulant matrix. Therefore, its elements satisfy

$$[\mathbf{H}]_{m,n} = h[(m-n)] = h[(m-n) \text{ modulo } N], 0 \le m, n \le N-1 \tag{6}$$

The basis vectors of the unitary DFT are columns of $\mathbf{F}^{*^T} = \mathbf{F}^*$, where $(\cdot)^*$ denotes the complex conjugate. So,

$$\phi_k = \left\{ \frac{1}{\sqrt{N}} e^{2\pi i k n/N}, 0 \le n \le N-1 \right\}, k = 0, \dots, N-1. \tag{7}$$

Now consider,

$$[\mathbf{H}\phi_k]_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h[m-n] e^{2\pi ikn/N}. \tag{8}$$

Writing $l = m - n$ and rearranging we have,

$$[\mathbf{H}\phi_k]_m = \frac{1}{\sqrt{N}} e^{2\pi ikm/N} \left[\sum_{l=0}^{N-1} h[l] e^{-2\pi ikl/N} + \sum_{l=-N+m+1}^{-1} h[l] e^{-2\pi ikl/N} + \sum_{l=m+1}^{N-1} h[l] e^{-2\pi ikl/N} \right]. \tag{9}$$

Using equation 6 and that $e^{2\pi il/N} = e^{-2\pi i(N-l)/N}$, since $e^{2\pi iN/N} = 1$, the second and third terms in the brackets in equation 9 cancel, thus we obtain the eigenvalue equation

$$[\mathbf{H}\phi_k]_m = \lambda_k \phi_k[m] \tag{10}$$

or

$$\mathbf{H}\phi_k = \lambda_k \phi_k \tag{11}$$

where λ_k, the eigenvalues of $\mathbf{H}$, are defined as

$$\lambda_k \triangleq \sum_{l=0}^{N-1} h[l] e^{-2\pi ikl/N}, 0 \leq k \leq N-1 \tag{12}$$

This is simply the DFT of the first column of **H**.[5] □

We note that a matrix is circulant if each row can be obtained as a shift, with circular wrap-around, of the previous row. It is clear that L_D is circulant, thus establishing a relationship between Fourier analysis and the Laplacian.

2. Laplacian Operator

With this connection between Fourier analysis and the Laplacian, we now consider the Laplacian Operator in a general setting.

2.1 Continuous Laplace-Beltrami Operator

We assume a compact Riemannian manifold $(\mathcal{M}, g)$ of dimension m, where $\mathcal{M}$ is a connected manifold and real-differentiable in C^∞. The function g defines for each point $p \in \mathcal{M}$ the inner product of the tangent space $T_p\mathcal{M}$. The union of all tangent spaces, $\bigcup_{p\in\mathcal{M}} T_p\mathcal{M} = T\mathcal{M}$. An m-dimensional manifold is a topological space, that locally resembles the Euclidean space $\mathbb{R}^m$. If the manifold $\mathcal{M}$ has a boundary $B = \partial\mathcal{M}$, it is assumed that $\mathcal{M}$ is oriented and that C^∞ also applies boundary B. Further, the outward unit normal vector field on B is denoted by u_n. We consider the real-valued function f with $f \in L^2(\mathcal{M}, g)$ and $f \in C^k$ with $k \geq 2$. The directional derivative of f at $p \in \mathcal{M}$ for each $\zeta \in T_p\mathcal{M}$ is denoted by ζf. The gradient of f is the vector field on $\mathcal{M}$ with

$$\text{grad} f = \zeta f, \forall \zeta \in T\mathcal{M}. \tag{13}$$

Now, let X, Y be a vector fields, which are in C^k, with $k \geq 1$, and let $\nabla_\zeta X$ be the covariant derivatives of X with respect to ζ for all $p \in \mathcal{M}$ and $\zeta \in T_p\mathcal{M}$. ∇_ζ satisfies $\nabla_\zeta(X+Y) = \nabla_\zeta X + \nabla_\zeta Y$ and $\nabla_\zeta(fX) = (\zeta f)X(p) + f(p)\nabla_\zeta X$. So, the divergence of X is defined by

$$\text{div} X = \text{trace}(\nabla_\zeta X), \tag{14}$$

where ζ ranges over $T_p\mathcal{M}$. Thus,

Definition 2.1. The Laplace-Beltrami Operator, Δ, is defined as

$$\Delta f = \mathrm{div}(\mathrm{grad} f). \tag{15}$$

If the manifold $\mathcal{M}$ possesses a boundary B, the following Boundary Conditions (BC) can be applied:

$$\text{The Dirichlet BC: } f = 0, \text{ on } \partial\mathcal{M} \tag{16}$$

which imposes that the Laplacian acts on those functions which vanish on the boundary.

$$\text{The Neumann BC: } \frac{\partial f}{\partial u_n} = u_n \cdot \Delta f = 0, \text{ on } \partial\mathcal{M} \tag{17}$$

which for manifolds correspond to the normal derivatives vanishing on the boundary.
The Δ operator has a number of properties that allow for analyzing harmonic basis functions by solving the Laplacian eigenvalue problem $\Delta\phi_i = \lambda_i\phi_i$ which include $\Delta f = 0$ for constant f, symmetry, local support, linear precision, maximum principle, and positive semi-definiteness.[3] Further, the set of all eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ defines the spectrum of $\mathcal{M}$,

$$spec(\mathcal{M}) = \{\lambda_i\}_{i=1}^{\infty} = \{0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_\infty\} \tag{18}$$

with $\lim_{i\to\infty} \lambda \to \infty$. The set of eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ forms an orthonormal basis and spectral analysis of functions defined on $\mathcal{M}$. As shown in Lemma 1.1, the eigensystem of the Laplace-Beltrami Operator can be considered as a basis of a generalized Fourier analysis on $\mathcal{M}$.
So, for any point p_i on $\mathcal{M}$, an approximation for Δ can be given be the curvilinear integral,

$$\Delta f(p_i) = \frac{1}{|\Gamma|}\int_{p\in\Gamma} (f(p_i) - f(p)\mathrm{d}p \tag{19}$$

where Γ is a closed simple curve on $\mathcal{M}$ surrounding points p_i, and $|\Gamma|$ is the length of Γ.
Further, specifically for a function $f : \mathbb{R} \to \mathbb{R}$, it is precisely the second derivative, for which we have the difference formula

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{20}$$

If we discretize the real line by dyadic points that can be written as $k/2^n$, for $k \in \mathbb{Z}$, $n \in \mathbb{N}$,

$$f''[x] = \lim_{n\to\infty} \frac{f[x+\frac{1}{2^n}] - 2f[x] + f[x-\frac{1}{2^n}]}{(\frac{1}{2^n})^2}. \tag{21}$$

2.2 Discrete Graph Laplacian

We now define the graph setting where we let $G = G(V, E)$ denote a graph. $V = \{x_i\}$ denotes the vertex set where we assume a finite graph, that is, the size of the vertex set is finite, $|V| = N < \infty$. The edge set, E, consists of ordered pairs where if there is an edge between points $x, y \in V$ we write $x \sim y$. So, $E = \{(x, y) : x, y \in V \text{ and } x \sim y\}$. Further, we define the degree of $x \in V$, d_x, to be the number of edges connected to point x. We consider functions defined on the vertex set $f : V \to \mathbb{R}$, $x_n \mapsto f[x_n]$.
If we consider equation 21, each vertex having an edge connecting it to its two closets neighbors, then we see it is the sum of all the differences of $f(x)$ with f evaluated at all it's neighbors. So, we write the Laplacian on a graph point-wise on a function $f : V \to \mathbb{R}$ as

$$\Delta_G f(x) = \sum_{x\sim y} f[x] - f[y] \tag{22}$$

For finite graphs, $|V| < \infty$, we can express the Laplace Operator as a matrix.

Definition 2.2. Let $\boldsymbol{D}$ be the degree matrix, a diagonal $N \times N$ matrix, be defined as

$$\boldsymbol{D} = diag(d_x).$$

Let $\boldsymbol{A}$ be the adjacency matrix, an $N \times N$ matrix, be defined as

$$\boldsymbol{A}(i,j) = \begin{cases} 1, & \text{if } x_i \sim x_j \\ 0, & \text{otherwise.} \end{cases}$$

So, the unnormalized Graph Laplacian is defined as

$$\boldsymbol{L}_G = \boldsymbol{D} - \boldsymbol{A} \tag{23}$$

or

$$\boldsymbol{L}_G(i,j) = \begin{cases} d_{x_i}, & \text{if } i = j \\ -1, & \text{if } x_i \sim x_j \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Further, the normalized Graph Laplacian is defined as

$$\boldsymbol{\mathcal{L}}_G = \boldsymbol{I}_N - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{1/2} = \boldsymbol{D}^{-1/2}\boldsymbol{L}_G\boldsymbol{D}^{1/2} \tag{25}$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix.

3. Graph Fourier Transform

With the graph setting and $\boldsymbol{L}_G$ defined, we now combine this with classical Fourier analysis to analyze the Graph Fourier Transform. We let functions f defined on a graph G to be written as vectors $f \in \mathbb{R}^N$, where $f[i]$ for $i = \{0, \ldots, N-1\}$, is the value of function f evaluated at vertex v_i. Further, we let $f \in \ell^2(G)$ and say $\|f\|_{\ell^2} = \left(\sum_{i=0}^{N-1} |f[i]|^2\right)^{1/2}$. So,

Definition 3.1. Given graph, G, and its Laplacian, $\boldsymbol{L}_G$, with spectrum $\sigma(\boldsymbol{L}_G) = \{\lambda_k\}_{k=0}^{N-1}$ and eigenvectors $\{\varphi_k\}_{k=0}^{N-1}$, the Graph Fourier Transform of $f : V \to \mathbb{C}$ and $f \in \ell^2(G)$ is defined by

$$\widehat{f}[\lambda_k] = \langle f, \varphi_k \rangle = \sum_{n=1}^{N} f[n]\varphi_k^*[n]. \tag{26}$$

We note that the Graph Fourier Transform is only defined on values of the spectrum of $\boldsymbol{L}_G$, $\sigma(\boldsymbol{L}_G)$. So $\widehat{f}$ will not be well-defined when λ_k has multiplicities greater than one. Further,

Definition 3.2. The Graph Inverse Fourier Transform is defined by

$$\widehat{f}[n] = \sum_{n=0}^{N-1} \widehat{f}[\lambda_k]\varphi_k[n]. \tag{27}$$

If we consider a function f and its Graph Fourier Transform $\widehat{f}$ as $N \times 1$ vectors, then equations 26 and 27 can be written as $\widehat{f} = \boldsymbol{\Phi}^* f$ and $f = \boldsymbol{\Phi}\widehat{f}$, respectively. We now see,

Proposition 3.1. *For any $f, g : V \to \mathbb{R}$, then Parseval's relation holds for the Graph Fourier Transform. That is, $\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$. Further, this implies,*

$$\|f\|_{\ell_2}^2 = \sum_{n=1}^{N} |f[x_n]|^2 = \sum_{\ell=0}^{N-1} |\widehat{f}[\lambda_l]|^2 = \left\|\widehat{f}\right\|_{\ell^2}^2. \tag{28}$$

Proof. If we consider $\boldsymbol{\Phi}$, the unitary matrix, then we see

$$\langle \widehat{f}, \widehat{g} \rangle = \widehat{f}^* \widehat{g} = (\boldsymbol{\Phi}^* f)^* \boldsymbol{\Phi}^* g = f^* \boldsymbol{\Phi}\boldsymbol{\Phi}^* g = f^* g = \langle f, g \rangle. \tag{29}$$

From this it is clear that $\|f\|_{\ell_2}^2 = \left\|\widehat{f}\right\|_{\ell^2}^2$.[7] □

4. Operations on the Graph Fourier Transform

4.1 Graph Convolution

In the classical setting we have,

Definition 4.1. For $f, g \in L^2(\mathbb{R})$, we define convolution as

$$(f * g)(t) = \int_{\mathbb{R}} f(u) g(t-u) \mathrm{d}u. \tag{30}$$

However, without a clear analogue of translation in the graph setting we make note of the convolution property $\widehat{(f * g)}(\gamma) = \widehat{f}(\gamma)\widehat{g}(\gamma)$. So by equation 27 we have,

Definition 4.2. For $f, g : V \to \mathbb{R}$, we define the Graph Convolution of f and g as

$$(f * g)[n] = \sum_{l=0}^{N-1} \widehat{f}[\lambda_l]\widehat{g}[\lambda_l]\varphi_l[n]. \tag{31}$$

So we see the following properties,

Proposition 4.1. *For $f, g, h : V \to \mathbb{R}$ and $\alpha \in \mathbb{R}$,*

1. $\widehat{f * g} = \widehat{f}\widehat{g}$
2. $\alpha(f * g) = (\alpha f) * g = f * (\alpha g)$
3. $f * g = g * f$
4. $f * (g + h) = f * g + f * h$
5. $(f * g) * h = f * (g * h)$
6. *Defining $g_0 \in \mathbb{R}$ by $g_0[n] = \sum_{l=0}^{N-1} \varphi_l[n]$. Then g_0 is an identity for the Graph Convolution such that $f * g_0 = f$*
7. $\boldsymbol{L}_G(f * g) = (\boldsymbol{L}_G f) * g = f * (\boldsymbol{L}_G g)$
8. $\sum_{n=1}^{N} (f * g)[n] = \sqrt{N}\widehat{f}[0]\widehat{g}[0] = \frac{1}{\sqrt{N}} \left[\sum_{n=1}^{N} f[n]\right] \left[\sum_{n=1}^{N} g[n]\right]$

The proof of these properties are direct results of Definition 4.2.[7]
Further, we can express Graph Convolution as a matrix operation,

$$f * g = \widehat{g}(\boldsymbol{L}_G) f = \boldsymbol{\Phi} \begin{bmatrix} \widehat{g}(\lambda_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \widehat{g}(\lambda_{N-1}) \end{bmatrix} \boldsymbol{\Phi}^* f. \tag{32}$$

4.2 Graph Modulation

Using the Euclidean space setting, where modulation of a function is multiplication of a Laplacian eigenfunction, we have

Definition 4.3. For any $k = 0, 1, \dots, N-1$, the Graph Modulation Operator, $\boldsymbol{M}_k : \mathbb{R}^N \to \mathbb{R}^N$, is defined as

$$(\boldsymbol{M}_k f)[n] = \sqrt{N} f[n] \varphi_k[n]. \tag{33}$$

Further, we can express the Graph Modulation Operator as a matrix,

$$\boldsymbol{M}_k = \begin{bmatrix} \varphi_k[1] & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \varphi_k[N] \end{bmatrix}. \tag{34}$$

We see that $\boldsymbol{M}_0$ is the identity operator, $\varphi_0[n] \equiv \frac{1}{\sqrt{N}}$, for all n. In the continuous setting, a modulation represents a translation in the Fourier domain, $\widehat{\boldsymbol{M}_\tau f}(\gamma) = \widehat{f}(\gamma - \tau), \forall \gamma \in \mathbb{R}$. However, this property does not hold true in the graph setting because of the discrete nature. Instead we have that if $\widehat{g}(l) = \delta_0(\lambda_l)$, then

$$\begin{aligned} \widehat{\boldsymbol{M}_\tau g}[\lambda_l] &= \sum_{n=l}^{N} (\boldsymbol{M}_\tau g)[n])\boldsymbol{\Phi}^*[n] \\ &= \sum_{n=l}^{N} \sqrt{N}\boldsymbol{\Phi}[n]\frac{1}{\sqrt{N}}\boldsymbol{\Phi}^*[n] \\ &= \delta_0[\lambda_l - \lambda_\tau] \\ &= \begin{cases} 1, & \text{if } \lambda_l = \lambda_\tau \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{35}$$

So, M_τ maps the constant component of $f \in \mathbb{R}$ to $\widehat{f}(0)\varphi_\tau$.

4.3 Graph Translation

In the continuous setting we have,

Definition 4.4. For $f \in L^2(\mathbb{R})$, the Translation Operator, τ_u, is defined as

$$(\tau_u f)(t) = f(t-u) = (f * \delta_u)(t) = \int_{\mathbb{R}} \widehat{f}(k)\widehat{\delta}(k)\varphi_k(t)\mathrm{d}k = \int_{\mathbb{R}} \widehat{f}(k)\varphi_k^*(u)\varphi_k(t)\mathrm{d}k \tag{36}$$

since $\widehat{\delta}_u(k) = \int_{\mathbb{R}} \delta_u(x)\varphi_k^*(x)\mathrm{d}x\varphi_k(u)$, where $\delta_u(x) = \begin{cases} 1, & \text{if } x = u \\ 0, & \text{if } x \neq u. \end{cases}$

So, in the graph setting,

Definition 4.5. For any $f : V \to \mathbb{R}$ we define the Graph Translation Operator, $\boldsymbol{T}_i : \mathbb{R}^N \to \mathbb{R}^N$, as

$$(\boldsymbol{T}_i f)[n] = \sqrt{N}(f * \delta_i)[n] = \sqrt{N}\sum_{l=0}^{N-1} \widehat{f}[\lambda_l]\varphi_l^*[i]\varphi_l[n] \tag{37}$$

that is, the graph convolution of the Dirac delta centered at the i^{th} vertex.

Further, we can express the Graph Translation Operator as a matrix operation,

$$\boldsymbol{T}_i f = \sqrt{N}\begin{bmatrix} \varphi_0^*[i]\varphi_0[1] & \varphi_1^*[i]\varphi_1[1] & \dots & \varphi_{N-1}^*[i]\varphi_{N-1}[1] \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_0^*[i]\varphi_0[N] & \varphi_1^*[i]\varphi_1[N] & \dots & \varphi_{N-1}^*[i]\varphi_{N-1}[N] \end{bmatrix} \begin{bmatrix} \widehat{f}[\lambda_0] \\ \vdots \\ \widehat{f}[\lambda_{N-1}] \end{bmatrix} = \sqrt{N}\boldsymbol{A}_i\boldsymbol{\Phi}^* f \tag{38}$$

Thus,

Proposition 4.2. *For any $f, g : V \to \mathbb{R}$ and $i, j \in \{1, 2, \dots, N\}$,*

1. $\boldsymbol{T}_i(f * g) = (\boldsymbol{T}_i f) * g = f * (\boldsymbol{T}_i g)$

2. $\boldsymbol{T}_i \boldsymbol{T}_j f = \boldsymbol{T}_j \boldsymbol{T}_i f$
3. $\sum_{n=1}^{N} (\boldsymbol{T}_i f)[n] = \sqrt{N} \hat{f}[0] = \sum_{n=1}^{N} f[n]$

Proof. Property 1 is clear from Definition 4.2.
For Property 2, we see,

$$\begin{aligned}
\boldsymbol{T}_i \boldsymbol{T}_j f[n] = & \sqrt{N} \textstyle\sum_{l=0}^{N-1} \widehat{\boldsymbol{T}_j f}[\lambda_l] \varphi_l^*[i] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} \sum_{m=1}^{N} (\boldsymbol{T}_j f)[m] \varphi_l^*[m] \varphi_l^*[i] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} \sum_{m=1}^{N} \sum_{k=0}^{N-1} \hat{f}[\lambda_k] \varphi_k^*[j] \varphi_k^*[m] \varphi_l^*[m] \varphi_l^*[i] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} \sum_{m=1}^{N} \sum_{k=0}^{N-1} \hat{f}[\lambda_k] \varphi_k^*[j] \delta_k(l) \varphi_l^*[i] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} \sum_{k=0}^{N-1} \hat{f}[\lambda_k] \varphi_k^*[j] \delta_k(l) \varphi_l^*[i] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} \hat{f}[\lambda_l] \varphi_l^*[j] \varphi_l^*[i] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} \hat{f}[\lambda_l] \varphi_l^*[i] \varphi_l^*[j] \varphi_l[n] \\
= & N \textstyle\sum_{l=0}^{N-1} (\widehat{f * \delta_i})[\lambda_l] \varphi_l^*[j] \varphi_l[n] \\
= & \sqrt{N} \boldsymbol{T}_j (f * \delta_i)[n] \\
= & \boldsymbol{T}_j \boldsymbol{T}_i f[n].
\end{aligned} \tag{39}$$

Finally, Property 3 is clear from Definitions 4.2 and 4.5.[1, 7] □

Further,

Corollary 4.3. *For any $f : V \to \mathbb{R}$ and $i, n \in \{1, 2, \ldots, N\}$,*

$$\boldsymbol{T}_i f[n] = \overline{(\boldsymbol{T}_n \overline{f})[i]} \tag{40}$$

Proof. Beginning from Definition 4.5 we have,

$$\begin{aligned}
(\boldsymbol{T}_i f)[n] = & \sqrt{N} \textstyle\sum_{l=0}^{N-1} \hat{f}[\lambda_l] \varphi_l^*[i] \varphi_l[n] \\
\overline{(\boldsymbol{T}_i f)[n]} = & \sqrt{N} \textstyle\sum_{l=0}^{N-1} \overline{\hat{f}[\lambda_l] \varphi_l^*[i] \varphi_l[n]} \\
= & \sqrt{N} \textstyle\sum_{l=0}^{N-1} \overline{\hat{f}}[\lambda_l] \varphi_l[i] \varphi_l^*[n] \\
= & (\boldsymbol{T}_n \overline{f})[i] \\
(\boldsymbol{T}_i f)[n] = & \overline{(\boldsymbol{T}_n \overline{f})[i]}
\end{aligned} \tag{41}$$

□

We note then that if the eigenvectors $\{\varphi_l\}_{l=0}^{N-1}$ are real-valued that,

$$(\boldsymbol{T}_i f)[n] = (\boldsymbol{T}_n f)[i] \tag{42}$$

From these properties we see that the Graph Translation Operator is distributive with convolution and that the translation operator commutes among other translation operators. However, many other properties that hold for the classical translation operator do not hold for the Graph Translation Operator. For example, in general, $\boldsymbol{T}_i \boldsymbol{T}_j \neq \boldsymbol{T}_{i+j}$. This is only true for shift-invariant graphs, that is, when the DFT basis vectors are exactly the the Graph Laplacian eigenvectors. Then,

$$\boldsymbol{T}_i \boldsymbol{T}_j = \boldsymbol{T}_{[((i-1)+(j-1)) \bmod N]+1}, \forall i, j \in \{1, 2, \ldots, N\}. \tag{43}$$

Further, the Graph Translation Operator is not isometric, that is, $\|\boldsymbol{T}_i f\|_2 \neq \|f\|_2$. However, we can state the following bounds on $\|\boldsymbol{T}_i f\|_2$,

Theorem 4.4. *For any $f : V \to \mathbb{R}$,*

$$|\widehat{f}(0)| \leq \|\boldsymbol{T}_i f\|_2 \leq \sqrt{N} \max_{l\in\{0,1,\dots,N-1\}} |\varphi_l[i]| \, \|f\|_2 \leq \sqrt{N} \max_{l\in\{0,1,\dots,N-1\}} \|\varphi_l\|_\infty \, \|f\|_2 \tag{44}$$

Proof. Beginning with Definition 4.5,

$$\begin{aligned}
\|\boldsymbol{T}_i f\|_2^2 &= N\sum_{n=1}^{N}\left(\sum_{l=0}^{N-1}\widehat{f}[\lambda_l]\varphi_l^*[i]\varphi_l[n]\right)^2 \\
&= N\sum_{l=0}^{N-1}\sum_{k=0}^{N-1}\widehat{f}[\lambda_l]\widehat{f}[\lambda_k]\varphi_l^*[i]\varphi_k^*[i]\sum_{n=1}^{N}\varphi_l[n]\varphi_k[n] \\
&= N\sum_{l=0}^{N-1}\sum_{k=0}^{N-1}\widehat{f}[\lambda_l]\widehat{f}[\lambda_k]\varphi_l^*[i]\varphi_k^*[i]\sum_{n=1}^{N}\delta_k(l) && (45)\\
&= N\sum_{l=0}^{N-1}|\widehat{f}[\lambda_l]|^2|\varphi_l^*[i]|^2 \\
&\leq N\max_{l\in\{0,1,\dots,N-1\}}|\varphi_l[i]|^2\,\|f\|_2^2 && (46)\\
&\leq N\max_{l\in\{0,1,\dots,N-1\}}\|\varphi_l\|_\infty^2\,\|f\|_2^2 && (47)
\end{aligned}$$

where equality in equation 45 comes from the fact that $\{\varphi_l\}_{l=0}^{N-1}$ is an orthonormal basis and can choose the eigenvectors to be real-valued. Further,

$$\begin{aligned}
\|\boldsymbol{T}_i f\|_2^2 &= N\sum_{l=0}^{N-1}|\widehat{f}[\lambda_l]|^2|\varphi_l^*[i]|^2 \\
&\geq N|\widehat{f}[\lambda_0]|^2|\varphi_0^*[i]|^2 \\
&= N|\widehat{f}[0]|^2\left|\frac{1}{\sqrt{N}}\right|^2 && (48)\\
&= |\widehat{f}[0]|^2 && (49)
\end{aligned}$$

where equation 48 comes from the fact that $\varphi_0[n] = 1/\sqrt{N}$.
Taking the square-roots of equations 46, 47, and 49 we obtain our bounds.[7] □

In addition we have,

Proposition 4.5. *In general, the Graph Translation Operator, $\boldsymbol{T}_i$, is not injective and therefore invertible.*

Proof. Let us suppose that there exists some $k \in \{1,2,\dots,N-1\}$ and $1 \in \{1,2,\dots,N\}$, for which $\varphi_k[i] = 0$. Then if $f = \varphi_k$, we have $\widehat{f}[\lambda_l] = \delta_k(l)$ and

$$\boldsymbol{T}_i f[n] = \sqrt{N}\sum_{l=0}^{N-1}\widehat{f}[\lambda_l]\varphi_l^*[i]\varphi_l^*[n] = \sqrt{N}\varphi_l^*[i]\varphi_l^*[n] = 0 \tag{50}$$

for all $n = 1,2,\dots,N$. Therefore, there exists a nonzero $f : V \to \mathbb{R}$ such that $T_i f = 0$. So, we see $\boldsymbol{T}_i$ is not injective and therefore not invertible.[1] □

However,

Theorem 4.6. *If $\varphi_k[i] \neq 0$ for all $k = 1,\dots,N-1$, then for $f : V \to \mathbb{R}$, the Graph Translation Operator $\boldsymbol{T}_i$ is invertible and its inverse is given by*

$$\boldsymbol{T}_i^{-1} f = \frac{1}{\sqrt{N}}\boldsymbol{\Phi}\begin{bmatrix}\varphi_0^*[1]\varphi_0[i]^{-1} & \varphi_0^*[2]\varphi_0[i]^{-1} & \dots & \varphi_0^*[N]\varphi_0[i]^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{N-1}^*[1]\varphi_{N-1}[i]^{-1} & \varphi_{N-1}^*[2]\varphi_{N-1}[i]^{-1} & \dots & \varphi_{N-1}^*[N]\varphi_{N-1}[i]^{-1}\end{bmatrix}\begin{bmatrix}f[0]\\ \vdots \\ f[N-1]\end{bmatrix} = \frac{1}{\sqrt{N}}\boldsymbol{\Phi}\boldsymbol{A}^{-1}f. \tag{51}$$

Proof. Using $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ given by equations 38 and 51, respectively, we see

$$\begin{aligned} \boldsymbol{A}_i\boldsymbol{A}_i^{-1}(n,m) &= \sum_{k=1}^{N} \boldsymbol{A}_i(n,k)\boldsymbol{A}_i^{-1}(k,m) \\ &= \sum_{k=0}^{N-1} \varphi_k^*[i]\varphi_k[n]\varphi_k^*[m]\varphi_k^*[i]^{-1} \\ &= \sum_{k=0}^{N-1} \varphi_k[n]\varphi_k^*[m] \\ &= \delta_n[m] \end{aligned} \tag{52}$$

and

$$\begin{aligned} \boldsymbol{A}_i^{-1}\boldsymbol{A}_i(n,m) &= \sum_{k=1}^{N} \boldsymbol{A}_i^{-1}(n,k)\boldsymbol{A}_i(k,m) \\ &= \sum_{k=0}^{N-1} \varphi_{n-1}^*[k]\varphi_{n-1}^*[i]^{-1}\varphi_{m-1}[k]\varphi_{m-1}^*[i] \\ &= \varphi_{n-1}^*[i]^{-1}\varphi_{m-1}^*[i]\sum_{k=1}^{N} \varphi_{n-1}^*[k]\varphi_{m-1}[k] \\ &= \varphi_{n-1}^*[i]^{-1}\varphi_{m-1}^*[i]\delta_n[m] \end{aligned} \tag{53}$$

so we have that $\boldsymbol{A}_i^{-1}\boldsymbol{A}_i = \boldsymbol{A}_i\boldsymbol{A}_i^{-1} = \boldsymbol{I}_N$. Thus,

$$\boldsymbol{T}_i\boldsymbol{T}_i^{-1} = \boldsymbol{A}_i\boldsymbol{\Phi}^*\boldsymbol{\Phi}\boldsymbol{A}_i^{-1} = \boldsymbol{I}_N = \boldsymbol{\Phi}\boldsymbol{A}_i^{-1}\boldsymbol{A}_i\boldsymbol{\Phi}^* = \boldsymbol{T}_i^{-1}\boldsymbol{T}_i \tag{54}$$

by the orthonormality of $\boldsymbol{\Phi}$.[1] □

5. Mesh Formulation and Application

We now consider an alternative formulation of the Laplacian on meshes where, in application, sensors located on an arbitrary surface, can be discretely represented as a triangular mesh in $\mathbb{R}^3$. A mesh $M = \{V, E, \mathcal{T}\}$, consists of vertices $v \in V$, edges $e \in E$, and triangles $t \in \mathcal{T}$ where each $v_i \in \mathbb{R}^3$ represents a sensor. The neighborhood N_i for vertex $v_i \in V$ is defined as $N_i = \{v_x \in V : e_{ix} \in E\}$. In this formulation we note angles α_{ij} and β_{ij} are located opposed to edge e_{ij}, connecting vertices v_i and V_j. Further, triangles t_a and t_b are defined by vertices (v_i, v_j, v_k) and (v_i, v_j, v_l), respectively, that share the edge e_{ij}. These mesh components are shown in Figure 1.

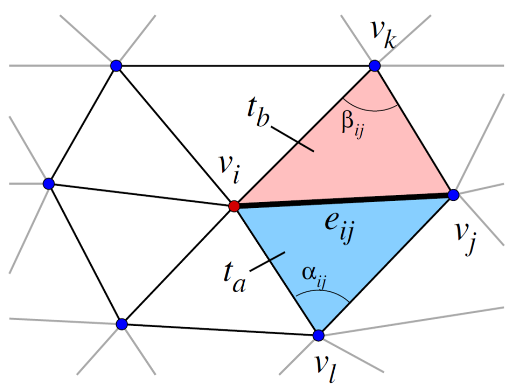

Figure 1: *Discrete representation of multi-sensor arrays on an arbitrary surface as a mesh with its components.[3]*

So, extending the approximation from equation 19,

Definition 5.1. For a function $f \in \mathbb{R}^N$ defined over all $v_i \in V$, $f : v_i \to \mathbb{R}$, with $i = \{1, \ldots, |V|\}$, a discretization of the continuous Laplace-Beltrami Operator on a mesh is defined as,

$$\Delta_M f_i = b_i^{-1} \sum_{x \in N_i} w(i,x)(f_i - f_x) \tag{55}$$

with weighing function $w(i,x)$ for edges $e_{ix} \in E$ and normalization coefficient b_i for vertex v_i.

This can be expressed in matrix form such that

$$\Delta_M f = -\boldsymbol{L}_M f \tag{56}$$

where the Laplacian matrix $\boldsymbol{L}_M$ can be expressed as

$$\boldsymbol{L}_M = \boldsymbol{B}^{-1}\boldsymbol{S} \tag{57}$$

where

$$\boldsymbol{B}^{-1} = \begin{cases} b_i^{-1}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{58}$$

and

$$\boldsymbol{S} = \begin{cases} \sum_{x \in N_i} w(i,x), & \text{if } i = j \\ -w(i,x), & \text{if } e_{ix} \in E \\ 0, & \text{otherwise.} \end{cases} \tag{59}$$

We call $\boldsymbol{B}$ and $\boldsymbol{S}$ the mass and stiffness matrices, respectively.
In a graph-theoretic approach, using $w(i,x) = b_i^{-1} = 1$ we obtain $\boldsymbol{L}_D$ from equation 5. Alternatively, using a geometric discretization through the Finite Element Method (FEM), we can compute $\boldsymbol{B}$ from equation 60 by,

$$\boldsymbol{B}(i,j) = \begin{cases} (\sum_{t \in \mathcal{T}_i} |t|)/6, & \text{if } i = j \\ (|t_a|+|t_b|)/12, & \text{if } e_{ij} \in E \\ 0, & \text{otherwise} \end{cases} \tag{60}$$

where $\mathcal{T}_i = \{t_x \in \mathcal{T} : v_i \in t_x\}$, the set of triangles sharing the vertex v_i.
Further, using a geometric discretization derived by minimizing the Dirichlet energy for a triangulated mesh, we can compute $\boldsymbol{S}$ from equation 59 with

$$w(i,x) = \frac{1}{2}(\cot(\alpha_{ix}) + \cot(\beta_{ix})). \tag{61}$$

Note that for edges on the boundary of the mesh, the $\cot(\beta_{ix}))$ term is omitted, leading to the Neumann BC.
This formulation allows for the geometric relationship between sensors to be taken into consideration. This has found application in analyzing electroencephalograms (EEG), a neuro-imaging method that places an array of electrodes at different positions on the scalp. Being able to consider the geometric relationship between sensors and their data may lead to better spatial understanding of the EEG signal and their physiological representation of neurological activity.[3, 6]

6. Conclusion

We have presented several extensions of classical Fourier analysis techniques in a graph setting noting some properties and operations that are equivalent and others that differ. Further analysis and exposition on the Graph Convolution, Modulation, and Translation Operators have been done but are not presented here.[1, 7] Finally, other classical Fourier analysis results have been studied and extended to the graph setting including the Uncertainty Principle [2], the Wavelet Transform [4], and Fourier Frames [7].

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